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Research projects for graduate students

Introduction The area of research of Y.P. is partial differential equations. A large part of his research focuses on the study of positive solutions of second-order elliptic and parabolic equations. This topic is closely related to the study of the structure and of the asymptotic behavior of solutions of linear and nonlinear elliptic and parabolic equations which appear in many branches of mathematics. For instance, positive solutions play an important role in some problems in differential geometry, in spectral theory of Schrödinger operators, in many questions in applied mathematics, and in some problems in Markov process theory. The focus of the research is on representation theorems, Green's functions and heat kernels estimates, Liouville theorems, variational principles, and removable singularities.

Scientific background Let P be a linear second-order elliptic operator which is defined in a subdomain Ω of a noncompact Riemannian manifold $\mathcal{M}$ of dimension d. Denote the cone of all positive solutions of the equation $Pu = 0$ in Ω by $\mathcal{C}_P(\Omega)$. The *generalized principal eigenvalue* with respect to a potential V is given by $\lambda_0(P, \Omega, V) := \sup\{\lambda \in \mathbb{R} \mid \mathcal{C}_{P-\lambda V}(\Omega) \neq \emptyset\}$. Suppose that $\mathcal{C}_P(\Omega) \neq \emptyset$. If P admits (resp. does not admit) a positive minimal Green function $G_P^\Omega(x, y)$ in Ω, then P is said to be a *subcritical* (resp. *critical*) operator in Ω. If $\mathcal{C}_P(\Omega) = \emptyset$, then P is said to be *supercritical* in Ω. If P is critical in Ω, then $\dim \mathcal{C}_P(\Omega) = 1$, and $\varphi \in \mathcal{C}_P(\Omega)$ is called a *ground state* of the operator P in Ω.

1. Perturbation theory

Level: Ph. D.
Prerequisites: The basic courses in PDEs and Functional analysis.

Overview Various types of perturbations by a potential V related to positive solutions of the elliptic equation $Pu = 0$ have been extensively studied over the past two decades. A further step will be to develop perturbation theory for the parabolic operator $L := \partial_t + P$ along the lines of perturbation theory for elliptic operators.

2. Heat kernel

Level: M. Sc. or Ph. D.
Prerequisites: The basic courses in PDEs and Functional analysis.

Overview The large time behavior of the heat kernel $k_P^\Omega(x,y,t)$ of the parabolic operator L is closely related to criticality theory. In particular, it is known that $\lim_{t\to\infty} \mathrm{e}^{\lambda_0 t} k_P^\Omega(x,y,t)$ always exists, and the limit is positive if and only if $P-\lambda_0$ is positive-critical. How fast is this limit approached? Davies conjectured that

$$\lim_{t\to\infty} \frac{k_P^\Omega(x,y,t)}{k_P^\Omega(x_0,x_0,t)} = a(x,y)$$

exists and positive for all $x,y \in \Omega$. Moreover, it is conjectured that if the limit exists, then $a(\cdot,y) \in \mathcal{C}_{P-\lambda_0}(\Omega)$. We propose to study Davies' conjecture under the assumption $\dim \mathcal{C}_{P-\lambda_0}(\Omega) = 1$, and in particular in the null-critical (nonsymmetric) case.

The following is a closely related conjecture: Let P_+ and P_0 be respectively subcritical and critical operators in Ω. Then

$$\lim_{t\to\infty} \frac{k_{P_+}^\Omega(x,y,t)}{k_{P_0}^\Omega(x,y,t)} = 0$$

locally uniformly in $\Omega \times \Omega$.

3. Criticality theory for the p-Laplacian with potential term

Level: M. Sc. or Ph. D.
Prerequisites: The basic courses in PDEs and Functional analysis.

Overview The question when a Hardy type inequality can be improved has been attracted many mathematicians . K. Tintarev and Y.P. gave an answer to this question in the general case. In particular, a criticality theory for the p-Laplacian with a potential term was developed. We propose to investigate Liouville theorems and the removability of singularity for such equations, to study positive solutions of minimal growth, and to extend the results to more general quasilinear equations.

Textbooks

[1] D. Gilbarg and N. S. Trudinger, “Elliptic partial differential equations of second order”, Reprint of the 1998 edition. Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[2] G. M. Lieberman, “Second order parabolic differential equations”, World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

[3] M. H. Protter and H. F. Weinberger, “Maximum principles in differential equations”, Corrected reprint of the 1967 original. Springer-Verlag, New York, 1984.